Chapter 1

Four-parameter curves

For the basic problem of interpolating splines, two-parameter splines are ideal. However, when greater than G^2 continuity is needed, or for designing particular shapes with straight-to-curve transitions, a four-parameter family of primitive curves is appropriate. Where a curve from a two-parameter family is determined by its tangent angles at the endpoints, a four-parameter curve is determined by both tangent angles and curvatures.

Section 1.1 motivates the need for four parameters in more detail. Section 1.2 discusses a well-known four-parameter spline, the Minimum Variation Curve, and presents a general equation for the shape of the MVC primitive curve. Section 1.3 presents an approximation to the MVC, where curvature is defined as a cubic polynomial of arclength. This spline has generally very similar properties, but is computationally much more tractable and easier to analyze. Finally, Section 1.4 presents a technique for applying labels to control points representing different constraints, generalizing the idea of blended corner curves with G^2-continuity to a richer set of configurations.

1.1 Why four parameters?

Chapter **??** contained a strong argument in favor of a two-parameter family of curves for interpolating splines. Why, then, consider a four-parameter family? When are two parameters not adequate?

First, while the human visual system appears to be sensitive only to G^2-continuity, there are applications for which higher orders of continuity are important. For example, "reflection lines" resulting from the specular reflection of lines from a 3D surface show generally exhibit only C^{k-1} continuity when the underlying surface is G^k (see [4, p. 570] for a detailed discussion). Thus, to guarantee smooth reflection lines, the surface must be designed with a high order of continuity. Another application demanding higher orders continuity is the design of rail layouts for high speed trains. At the slower speeds characteristic of trains in the United States, G^2 continuity is adequate, and clothoidal segments are the usual design norm, but rest of the world there has seen recent attention on smoother curves [5].

Figure 1.1 shows an example of a simple closed curve (an ovoid shape with eccentricity 0.56199) drawn using both a G^2-continuous spline and a G^4 continuous spline. It is clear from the curvature plots that the curvature variation is smoother in the G^4 case, but the points of discontinuity of curvature variation are not directly visible to the eye.

Second, many shapes are not best represented as a simple interpolating spline. The outline of a letterform is not, in general, a smooth undulating curve (although such examples do exist, such as the lowercase 'c' in Figure **??**). In addition to sections of smooth curve, a letterform may also have sharp corners, sections of straight line, and, most importantly for this discussion, smooth transitions from straight lines to curved sections. A simple example is shown in Figure 1.2 (based on Fig. 4.10 in Moreton's Ph. D. thesis [6]), illustrating the problem of blending smooth corners between sections of straight line.

In font design, similar situations arise for bracketed serifs, and for shapes such as the letter U, where straight sections blend smoothly with curves. A typical example showing both situations is shown in Figure 1.3, a capital U from the author's font Cecco. The entire curve (with the exception of the sharp corners) is G^2 continuous, providing pleasing and smooth joins between the straight and curved sections.

A four-parameter curve family provides enough flexibility to meet such needs. It's clear that a single section of Euler spiral is not adequate for one of the corners in Figure 1.2, as the curvature must start out zero, increase to a

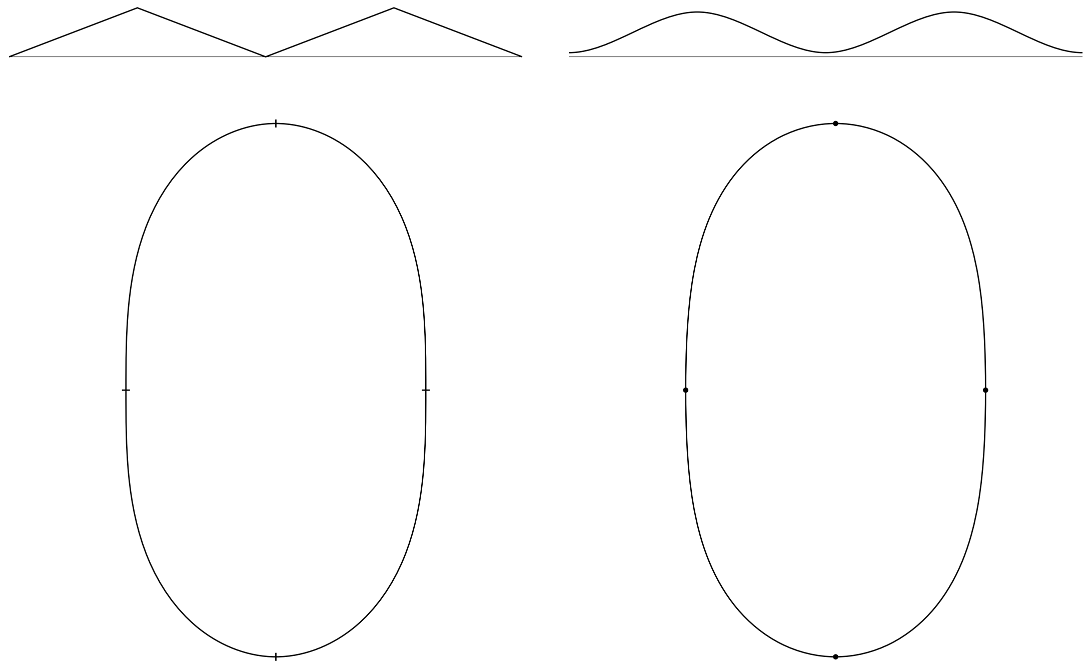

Figure 1.1: Eccentricity 0.56199 oval with G2 and G4 continuity.

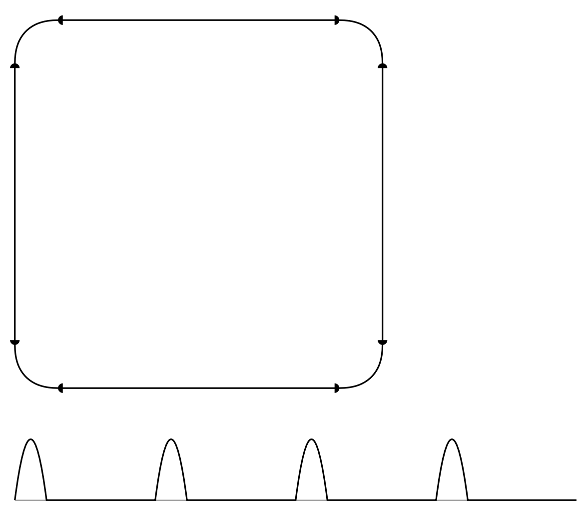

Figure 1.2: The suitcase corners problem.

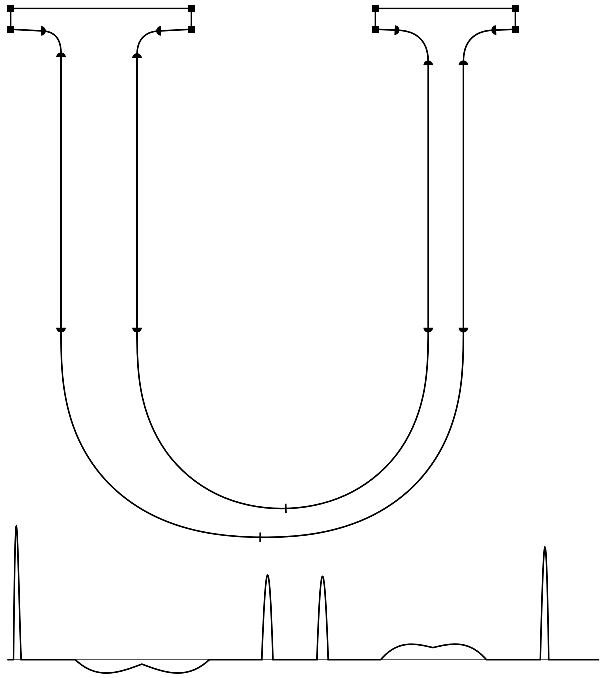

Figure 1.3: Bracketed serifs and straight-to-curve joins.

value large enough for the segment to turn by a right angle, then decrease again to zero.

Third, by annotating curves with additional constraints, it's possible to improve locality. In particular, one-way constraints can isolate sections of extreme changes of curvature, so that those sections do not influence nearby regions of gentle curvature change.

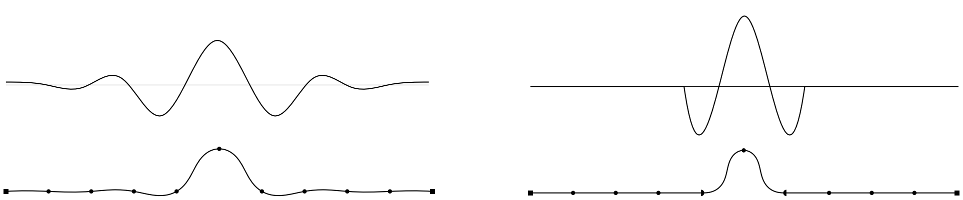

Figure 1.4: One-way constraints enforce locality.

For example, Figure 1.4 shows how adding one-way constraints keeps the variation of curvature localized to just two segments of the overall spline, as opposed to the infinite extent of a pure interpolating spline. The central point can be perturbed arbitrarily without affecting the segments beyond the one-way constraints. In this figure, the top plots show curvature as a function of arclength, and the bottom plots show the control points (with symbols representing the constraints) with the interpolating curve. Note also that the curve with more locality also requires considerably more curvature (and variation of curvature) to fit the points.

There are two examples of a four-parameter primitive curve, both straightforward generalizations of two-parameter counterparts. One is the Minimal Variation Curve, which minimizes the L2 norm of curvature variation. Another is the generalization of the Euler spiral so that curvature is an arbitrary cubic polynomial in arclength (the definitition of the Euler spiral is a linear relationship). Both of these splines are G^4-continous and use a primitive curve which has four parameters. In practice, these curves are quite similar, and one can easily be considered an approximation of the other.

The parameter space of a two-parameter spline can be understood as the tangent angles of the endpoints, relative to the chord. Given arbitrary endpoint tangents, a two-parameter curve family finds a smooth curve meeting those constraints. Similarly, a four-parameter curve family finds a smooth curve meeting given tangent and curvature values at the endpoints.

1.2 Minimum Variation Curve

The Minimum Variation Curve (MVC) is defined as the curve minimizing the L2 norm of the *variation* in curvature. Moreton's Ph. D. thesis [6] argued in favor of this spline to address known limitations of the MEC. In particular, the MEC lacks roundness, but since a circular arc has zero curvature variation, it is trivially the curve that minimizes the MVC cost functional when the input points are co-circular.

$$E_{MVC}[\kappa(s)] = \int_0^l (\kappa')^2 \; ds \tag{1.1}$$

Most presentations of the MVC, including Moreton's, use a numerical approach, approximating the curve with some other approximation (in Moreton's case, quintic polynomials) and using a technique such as conjugate gradient descent to minimize the cost functional. Gumhold's more recent approach [3] is similar, also using conjugate gradient descent. However, a variational approach analogous to that of the MEC produces exact differential equations for the curve. The simple Euler-Lagrange equation is inadequate for this task, as it only handles functionals written in terms of a first derivative. Since the MVC functional is in terms of the second derivative of θ, the more geneal Euler-Poisson equation is needed. Section 1.2.1 recaps the general technique from the calculus of variations, then Section 1.2.2 applies it to the MVC and gives the solution. One formulation is the impressively simple $\kappa'' = \lambda y$, which is pleasingly analogous to $\kappa = \lambda y$ as the general solution to the elastica (Equation **??**).

1.2.1 Euler-Poisson equation

Here we repeat the formulation of the general Euler-Poisson equation as given by Elsgolts [2]:

Let us investigate the extreme value of the functional

$$v[y(x)] = \int_{x_0}^{x_1} F(x, y(x), y'(x), \ldots, y^{(n)}(x)) \; dx, \tag{1.2}$$

where we consider the function F differentiable $n+2$ times with respect to all arguments and we assume that boundary conditions are of the form

$$y(x_0) = y_0,\ y'(x_0) = y_0',\ \ldots,\ y^{(n-1)}(x_0) = y_0^{(n-1)}; \tag{1.3}$$

$$y(x_1) = y_1,\ y'(x_1) = y_1',\ \ldots,\ y^{(n-1)}(x_1) = y_1^{(n-1)}, \tag{1.4}$$

i.e. at the boundary points are given the values not only of the function but also of its derivatives up to the order $n-1$ inclusive.

The function $y = y(x)$ which extremizes the functional given in Equation 1.2 must be a solution of the equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial F}{\partial y''} + \ldots + (-1)^n \frac{d^n}{dx^n}\frac{\partial F}{\partial y^{(n)}} = 0 \tag{1.5}$$

1.2.2 Euler-Poisson solution of MVC

We add a Lagrange multipliers to represent the constraint that the curve spans the endpoints. Adding the Lagrange multipliers and eliminating κ in favor of θ, we seek to minimize

$$E = \int_0^l (\theta'')^2 + \lambda_1 \sin\theta + \lambda_2 \cos\theta \; ds \tag{1.6}$$

To apply this equation to the MVC, we substitute $x = s$ and $y(x) = \theta(s)$, so that

$$F(x, y, y', y'') = (y'')^2 + \lambda_1 \sin y + \lambda_2 \cos y \tag{1.7}$$

Computing the relevant partial derivatives,

$$\frac{\partial F}{\partial y} = \lambda_1 \cos y - \lambda_2 \sin x \tag{1.8}$$

$$\frac{\partial F}{\partial y'} = 0 \tag{1.9}$$

$$\frac{\partial F}{\partial y''} = 2y'' \tag{1.10}$$

$$\frac{d^2}{dx^2}\frac{\partial F}{\partial y''} = 2y'''' \tag{1.11}$$

Applying 1.5 and reversing the variable substitution, we derive an ordinary differential equation for the MVC:

$$\lambda_1 \cos\theta - \lambda_2 \sin\theta + 2\theta'''' = 0 \tag{1.12}$$

Because at this point we are concerned only about the existence of the Lagrange multipliers, not their actual values, we can ignore the constant factor of $frac12$ and $-frac12$ for λ_1 and λ_2, respectively.

$$\theta'''' + \lambda_1 \cos\theta + \lambda_2 \sin\theta = 0 \tag{1.13}$$

Note the pleasing similarity to the elastica (MEC) equation:

$$\text{elastica:} \quad \theta'' + \lambda_1 \cos\theta + \lambda_2 \sin\theta = 0 \tag{1.14}$$

Also note that, as a special case when λ_1 and λ_2 become zero, all second order polynomial spirals $\kappa(s) = k_0 + k_1 s + k_2 s^2$ are solutions to the MVC equation, as the the third derivative of κ (the fourth derivative of θ) vanishes. These solutions are analogous to circular arcs $\kappa(s) = k_0$ as solutions of the elastica when the corresponding Lagrange multipliers become zero.

Again as in the MEC case, we can reformulate this equation solely in terms of first and higher derivatives of θ, also eliminating the Lagrange multipliers.

First, we differentiate Equation 1.13:

$$\theta^{(5)} + \theta'(-\lambda_1 \sin\theta + \lambda_2 \cos\theta) \tag{1.15}$$

To integrate Equation 1.13, we first multiply by θ':

$$\theta''''\theta' + \theta'(\lambda_1 \cos\theta + \lambda_2 \sin\theta) = 0 \tag{1.16}$$

This equation can readily be integrated:

$$\theta'''\theta' - \frac{(\theta'')^2}{2} + (\lambda_1 \sin\theta - \lambda_2 \cos\theta) + C = 0 \tag{1.17}$$

Combining with Equation 1.15,

$$\theta^{(5)} + \theta'''(\theta')^2 - \theta'(\theta'')^2/2 + C\theta' = 0 \tag{1.18}$$

Rewriting in terms of κ,

$$\kappa'''' + \kappa''\kappa^2 - \frac{\kappa(\kappa')^2}{2} + C\kappa = 0 \tag{1.19}$$

Note that, in cases where the length is not constrained, the constant C becomes zero. This follows from transversality, a standard technique in the calculus of variations [2, p. 345], and is analogous to the way the corresponding constant becomes zero in the case where the MEC is not length constrained. Then, the equation becomes:

$$\kappa'''' + \kappa''\kappa^2 - \tfrac{1}{2}\kappa(\kappa')^2 = 0 \tag{1.20}$$

The result of Equation 1.20 was also given by Brook et al [1]; their solution does not encompass additional length constraints (nonzero values of C).

Standard numerical techniques (such as Runge-Kutte integration) can be used to approximate the exact shape of the curve very precisely. The basic MVC spline can be defined as the curve which enforces G^4-continuity across control points, using the above-defined equation as a primitive between each pair of adjacent control points.

The MVC spline (ie the curve minimizing the MVC functional of Equation 1.1 that interpolates the control points) is based on the MVC primitive, in much the same way that the MEC spline is based on the rectangular elastica. Each segment is defined by four parameters. Across control points, G^4 continuity is enforced, and at endpoints, $\kappa' = 0$ and $\kappa'' = 0$. It's fairly easy to see why the latter conditions hold. At an endpoint, it is possible to add at least points lying on a circular arc of matching tangent and curvature. The resulting MVC spline must assign exactly that circular arc to the additional segments, as the value of the MVC functional itself for the arc is zero, and if there is any other curve with a lower value for the functional, then it would have been a better solution for the original problem. And since the MVC is G^4-continuous, then κ' and κ'' must match the values for a circular arc, which are zero. These endpoint conditions can also be derived straightforwardly (if abstractly) from the transversality condition.

There is also a simple equation for the MVC relating curvature and a Cartesian coordinate, analogous to $\kappa = \lambda y$ for the MEC case. To derive it, we first assume without loss of generality that $\lambda_1 = 0$; unlike the formulation in terms of curvature alone, the new equation will not be rotationally invariant. We then observe that $\frac{dy}{ds} = \sin\theta$. Substituting that into Equation 1.13, we get:

$$\theta'''' + \lambda_2 y' = 0 \tag{1.21}$$

Integrating (the constant of integration can be assumed zero, since that correponds to a translation of the curve), substituting $\kappa = \theta'$, and adjusting the constants so $\lambda = -\lambda 2$, the resulting simple formulation is:

$$\kappa'' = \lambda y \tag{1.22}$$

1.3 Polynomial spiral spline

Another approach to four-parameter splines is to generalize the Euler spiral to an implicit equation where curvature is a cubic polynomial function of arclength. The general equation is:

$$\kappa(s) = a + bs + cs^2 + ds^3 \tag{1.23}$$

Note that this curve family is a very good approximation to the MVC. Assuming small turning angles, y is nearly proportional to s, so substituting Equation 1.22, $\kappa'' \approx c_1 s + c_2$. Integrating twice (and bringing about two more constants of integration) yields Equation 1.23. This approximation is analogous to that of the Euler spiral approximating the MEC, but can be expected to be even closer, as the nonlinear terms affect only higher order derivatives of curvature.

As in the MVC, the corresponding spline can be defined in terms of enforcing G^4-continuity across control points. These are four constraints per control point. Since each segment requires four constraints to determine the four parameters of the polynomial, for an open curve there are four additional parameters, or two for each endpoint. One approach is to set tangent and curvature explicitly, but a more natural approach (in the interpolating spline framework) is to set set $\kappa' = 0$ and $\kappa'' = 0$ at the endpoints, as in the MVC. Note that, in this respect, the cubic polynomial spline is even more closely related to the MVC than the Euler spiral is to the MEC, because in the latter case the endpoint conditions differ; the natural end condition is constant curvature (a circular arc), while the natural endpoint condition for the MEC is zero curvature. In the four-parameter world, the MVC endpoint conditions make sense and there is no need to adjust them.

The cubic polynomial spiral spline was first proposed by Ohlin [7], by analogy of making the pentic (polynomial) spline nonlinear in the same way that the MEC is the nonlinear counterpart of the cubic spline. Although Ohlin did express the natural variational formulation of polynomial splines, and cited Mehlum's work indicating that the Euler spiral spline was a consistent (extensional) approximation to the MEC, he did not explicitly mention the MVC or state that this new spline is a good approximation to it.

The idea of a curve defined as curvature being a cubic polynomial function of arclength goes back even further. In 1936, Bloss proposed a railroad transition spiral in which the curvature is a simple cubic polynomial with respect to arclength [5]. The Euler spiral (or clothoid, as it is nearly universally known in that application domain) is popular in railroad track design. The linear change in curvature is desirably smooth on the interior of the segment, but the literature has also long recognized the limitations of smoothness at the joins between cothoid segments. These higher derivatives of curvature become more significant as train speeds increase. In particular, high speed train tracks are almost invariably banked (or "superelevated") so that the rail on the outside of the curve is higher than the inside, to a degree roughly proportional to the curvature. Thus, passengers of a train traversing the tracks experience a *roll acceleration* roughly equal to the *second* derivative of curvature. At the join of two Euler spiral segments, this roll acceleration is an impulse. Usually, the suspension of the train provides enough cushioning to avoid an unpleasant lurch, but curves with better continuity properties are clearly desirable for a maximally smooth, quiet ride. The Bloss transition spiral is a very early example of a curve with higher continuity, to avoid such impulses.

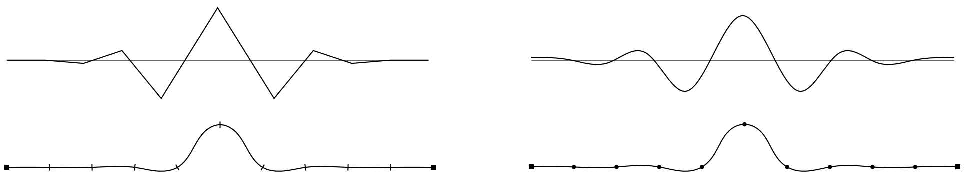

Figure 1.5: G^2 spline has better locality than G^4.

If G^4-continuity is smoother than G^2, then why not use it everywhere? Unfortunately, the additional smoothness comes at the cost of locality. As shown in Figure 1.5, the effects of perturbing a point ripple out almost twice as far as for the Euler spiral spline. Since, for most applications, the difference in smoothness is theoretical, rather than visible to the eye, it's better to enjoy the superior locality properties of the G^2-continuous spline.

1.4 Generalized constraints

The motivation for four-parameter splines showed a number of specific examples of blended corners, straight-to-curve transitions, and so on. This section presents a highly general technique that subsumes all these examples, and provides a broad palette of curve transitions to the designer.

A section of G^4-continuous spline has two free parameters at each endpoint. The natural MVC endpoint is to set the first and second derivative of curvature to zero, but it is equally valid to set these additional two parameters by constraining tangent and curvature at the endpoint.

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The basic concept is the "one-way" constraint, which annotates a point with a straight side and a curved side. In simple cases, the overall spline curve can be computed in two passes. First, solve the spline on the straight side of the constraint point, treating that point the same as an endpoint. If (as is the case for actual straight-to-curved transitions) that segment spline consists of only two points, then the result is a straight line, and the curvature is obviously zero. Then, solve the spline on the curved side of the constraint point, fixing both the tangent angle and the curvature at the constraint point. A spline based on a four-parameter primitive has enough degrees of freedom to satisfy these constraints.

Such a constraint is considered "one-way" because any perturbation on the curved side does not affect the shape of the curve on the straight side. For an ordinary interpolating spline, such a perturbation has an effect on the entire curve, although the amplitude dies out exponentially. Thus, isolating regions of high curvature inside explicitly annotated one-way constraints also improves the locality of the resulting spline.

However, the procedure describe above is not quite adequate to handle all cases. For example, what if the locality properties of the Euler spiral spline is desired for the inner segment? What about other edge cases, such as two one-way constraints in a row? Such cases do arise in designing shapes such as font outlines.

In the general approach, each segment between two adjacent control points is assigned a cubic polynomial spiral segment (i.e. as defined by Equation 1.23). Note that an Euler spiral spline fits into this framework; the parameters c and d are both zero. Equivalently, κ'' is constrained to be zero at both endpoints. For all types of internal points (other than corners, which are considered endpoints) are also constrained to have G^2-continuity, in other words, both tangent angle θ and curvature κ are constrained to be equal on both sides of a control point. Different points have additional constraints, depending on the type of point.

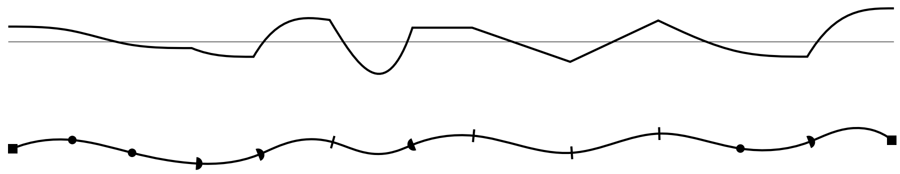

Figure 1.6: Examples of different kinds of constraint points.

The total number of degrees of freedom (and thus, number of constraints) is four times the number of segments. If each endpoint has two constraints and each internal point has four, then the total number of constraints is satisfied.

The complete list of point types, and associated mathematical constraints, is given in the table below.

Type of constraint	**Formula**		**Symbol**
Left endpoint		$\kappa'_r = 0$	
		$\kappa''_r = 0$	
Right endpoint		$\kappa'_l = 0$	
		$\kappa''_l = 0$	
G^2 curve point	$\theta_l = \theta_r$	$\kappa''_l = 0$	
	$\kappa_l = \kappa_r$	$\kappa''_r = 0$	
G^4 curve point	$\theta_l = \theta_r$	$\kappa'_l = \kappa'_r$	
	$\kappa_l = \kappa_r$	$\kappa''_l = \kappa''_r$	
One-way (terminal to curve)	$\theta_l = \theta_r$	$\kappa'_l = 0$	
	$\kappa_l = \kappa_r$	$\kappa''_l = 0$	
One-way (curve to terminal)	$\theta_l = \theta_r$	$\kappa'_r = 0$	
	$\kappa_l = \kappa_r$	$\kappa''_r = 0$	

For a given sequence of points with constraint annotations, the resulting spline is defined as the curve that is piecewise cubic polynomial spline, such that the tangent, curvature, and higher-order derivatives satify the constraints given in the table above for each point.

Figure 1.4 shows a sequence of somewhat arbitrary constraints, parallel with a curvature plot which shows the effect of the different types of constraints in the curvature domain. All segments between two G^2 curve points has linear curvature. All segments between a G^2 point and an endpoint (or between a G^2 point and the straight side of a one-way constraint) have constant curvature, i.e. the segment is a circular arc. All of the points have at least G^2 continuity, although if corner points were added, those would have only G^0.

Section **??** presents numerical techniques for satisfying such a system of constraints, and Chapter **??** discusses techniques for efficiently drawing the polynomial spline curves that result. With all these components in place, the spline is both highly flexible for drawing the types of shapes likely to arise in font design, and also entirely practical for interactive editing.

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